Bilinear form:

Defn (Bilinear form): V n-dim'l vector space over $\mathbb{R}$, $\langle\ ,\ \rangle$ is a bilinear form if

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R}.$$
$$(v, w) \mapsto \langle v, w \rangle.$$

$a \in \mathbb{R}$, $v, w \in V$. $v_1, v_2 \in V$.

① $\langle av, w\rangle = a\langle v, w\rangle$

$\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$

② $\langle w, av\rangle = a\langle w, v\rangle$

$\langle w, v_1 + v_2\rangle = \langle w, v_1\rangle + \langle w, v_2\rangle.$

Defn (Symmetric bilinear form) or symmetric form

$\langle v, w\rangle = \langle w, v\rangle$

Defn (skew symmetric form, alternating form

$\langle v, w\rangle = -\langle w, v\rangle$

Ex 1: Euclidean space $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\}$.

$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\langle v, w \rangle = \sum x_i y_i$. (standard inner product)

Angle between non zero v, w,

$$\cos\theta = \frac{\langle v, w \rangle}{\sqrt{\langle v, v \rangle \langle w, w \rangle}}$$

Ex 2: Minkowski space. (Lorentz form)

$$\langle v, w \rangle = x_1 y_1 + x_2 y_2 \cdots x_{n-1} y_{n-1} - x_n y_n.$$

Ex1, Ex2 are symmetric bilinear forms.

Gram matrix $B: v_1, \dots, v_n$ basis of V. $\langle , \rangle$ bilinear

$$(\langle v_i, v_j \rangle)_{n\times n} = G_{B, \langle , \rangle}$$

$\forall v \in V$, $v = \sum_{i=1}^{n} x_i v_i$, $w = \sum_{j=1}^{n} y_j v_j$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (\bar{v})_B$. $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (\bar{w})_B$

Prop: $\langle v, w \rangle = x^T \cdot G_{B, \langle , \rangle}\, y$

Prop: If $(w_1, \dots, w_n) = (v_1, \dots, v_n) \cdot P$

is another basis, then

Gram matrix under basis B': $(w_1, \dots, w_n)$

$G_{\langle , \rangle, B'} = P^T G_{\langle , \rangle, B} P$. ($\Leftarrow$ matrix congruence)

Pf: $P = (p_{ij})$, $w_j = \sum_i p_{ij} v_i$

$\langle w_j, w_k \rangle = \langle \sum_i p_{ij} v_i, \sum_l p_{lk} v_l \rangle$

$\Rightarrow$ $G_{\langle , \rangle, B'} = P^T G_{\langle , \rangle, B} P$

Defn: $A_1, A_2 \in M_n(\mathbb{R})$, congruent iff $\exists\, P \in GL(n, \mathbb{R})$,
s.t. $P^T A_1 P = A_2$ (相合)

Defn: (Isometry) $(V_1, \langle\,,\,\rangle_1)$, $(V_2, \langle\,,\,\rangle_2)$
isometric iff $\exists\, f: V_1 \to V_2$. s.t.

$$\langle v, w\rangle_1 = \langle f(v), f(w)\rangle_2 \quad \forall\, v, w \in V_1$$

(Such an f is called isometry)

Prop: $(V_1, \langle\,,\,\rangle_1)$, $(V_2, \langle\,,\,\rangle_2)$
are isometric iff the two Gram matrices are congruent.

Ex: $V = M_{m\times n}(\mathbb{R})$. $\langle\,,\,\rangle_F$ (Frobenius)

$$\langle A, B\rangle_F = Tr(A^T B)$$

$V, \langle\,\rangle_F$ isometric to $\mathbb{R}^{mn}$, $\langle\,,\,\rangle_{standard}$

Prop: $V, \langle \; \rangle$ symmetric iff the Gram matrices are symmetric ($A = A^T$)

From now on: $\langle , \rangle$ is symmetric.

Prop: (polarization identity) $\langle , \rangle$ symmetric, then

$$\langle v, w \rangle = \frac{1}{2}\left(\langle v+w, v+w \rangle - \langle v, v \rangle - \langle w, w \rangle\right)$$

pf: By expansion.

Thm: There exists basis of V, $\mathcal{B}$: $v_1 \ldots v_n$,
(Sylvester) such that $v = \sum x_i v_i$.

$$\langle v, v \rangle = x_1^2 + \cdots x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

p, q are determined by $\langle , \rangle$. ∴ uniqueness

or

$$G_{\langle , \rangle \mathcal{B}} = \begin{bmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{bmatrix}$$

pf: (Existence)

By polarization identity, if $\langle , \rangle$ is not zero, then $\exists\, v_1 \in V$, s.t. $\langle v_1, v_1 \rangle = a \neq 0$

Define $W = \{ w \in V \mid \langle w, v_1 \rangle = 0 \}$.

Claim: $W \oplus \mathbb{R} v_1 = V$.

(a) $W \cap \mathbb{R} v_1 = \{0\}$. If $c v_1 \in W$.

$\langle c v_1, v_1 \rangle = ca = 0 \Rightarrow c = 0$

(b) $\forall v \in V$, (Try to find c, s.t.

$v = w + c v_1$, $w \in W$

$\Leftrightarrow \langle v - c v_1, v_1 \rangle = 0$

$\Leftrightarrow \quad c = \dfrac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle}$)

Let $c = \dfrac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle}$

Then $v - c v_1 = w \in W$. $\Rightarrow v \in W + \mathbb{R} v_1$

$\dim W = n-1$, By induction on n.

W has basis $w_1 \cdots w_{n-1}$ s.t.

$$\langle w_i, w_j \rangle = \begin{cases} 0 & i \neq j \\ \pm 1, 0 & i = j \end{cases}$$

Let $w_0 = \dfrac{v_1}{\sqrt{|a|}} \Rightarrow \langle w_0, w_i \rangle = 0 \quad \forall i > 0$

$\langle w_0, w_0 \rangle = \pm 1$.

Rearrange the order of $w_0 \cdots w_{n-1} \Rightarrow$

Basis $v_1 \cdots v_n$, s.t. $\langle v_i, v_j \rangle$

$$= \begin{cases} 0 & i \neq j \\ 1 & 1 \le i = j \le p \\ -1 & p+1 \le i = j \le p+q \\ 0 & i = j \ge p+q+1 \end{cases}$$

(The same method for any field $\Rightarrow \exists$ basis s.t. Gram matrix is diagonal)

Uniqueness:

$W_1 = \text{span}(v_1, \dots v_p)$

$W_2 = \text{span}(v_{p+1} \dots v_{p+q})$

$W_3 = \text{span}(v_{p+q+1} \dots v_n)$

$V = W_1 \oplus W_2 \oplus W_3$

If under basis B': $v_1' \dots v_{p'}' \dots v_{q'}' \dots v_n$, s.t. $G_{C,B'} =$

$$\begin{bmatrix} I_{p'} & & \\ & -I_{q'} & \\ & & 0 \end{bmatrix}$$

then $V = W_1' \oplus W_2' \oplus W_3'$, and we claim

$\dim W_i = \dim W_i'$,

If $\dim W_1 > \dim W_1'$. then $\dim W_1 + \dim(W_2' \oplus W_3') > n$

$\Rightarrow W_1 \cap (W_2' \oplus W_3') \neq 0.$

$$\Rightarrow \exists\, v = \sum_{i=1}^{p} a_i v_i = \sum_{j=p'+1}^{n} b_j' v_j' \neq 0,$$

$$\Rightarrow \langle v, v \rangle = \sum_{i=1}^{p} a_i^2 > 0 \quad (\text{since not all } a_i = 0)$$

$$\langle v, v \rangle = -\sum_{j=p'+1}^{p'+q'} b_j'^2 \leq 0, \quad \text{Contradiction}$$

So $\dim W_1 \leq \dim W_1'$

For the same reason $\dim W_1 \geq \dim W_1'$

$p = p'$, and $q = q'$. $\Rightarrow (p, q, n-p-q)$

$= (p', q', n-p'-q')$ $\square$

$(p, q, n-p-q)$ is called the signature of $\langle\, ,\, \rangle$.

Sylvester Thm in terms of matrices:

Thm: For any symmetric real matrix A,

$\exists P$ invertible, s.t.

$$P^T A P = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

and $(p, q, n-p-q)$ uniquely determines the congruence class of A.

(signature of A)